

The Complexity of Grid Coloring

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Grid Coloring

Notation: If $n \in \mathbb{N}$ then $[n]$ is the set $\{1, \dots, n\}$.

Definition

$G_{n,m}$ is the grid $[n] \times [m]$.

1. $G_{n,m}$ is **c -colorable** if there is a c -coloring of $G_{n,m}$ such that no rectangle has all four corners the same color.
2. $\chi(G_{n,m})$ is the least c such that $G_{n,m}$ is c -colorable.

A FAILED 2-Coloring of $G_{4,4}$

<i>R</i>	<i>B</i>	<i>B</i>	<i>R</i>
<i>B</i>	<i>R</i>	<i>R</i>	<i>B</i>
<i>B</i>	<i>B</i>	<i>R</i>	<i>R</i>
<i>R</i>	<i>R</i>	<i>R</i>	<i>B</i>

A 2-Coloring of $G_{4,4}$

<i>R</i>	<i>B</i>	<i>B</i>	<i>R</i>
<i>B</i>	<i>R</i>	<i>R</i>	<i>B</i>
<i>B</i>	<i>B</i>	<i>R</i>	<i>R</i>
<i>R</i>	<i>B</i>	<i>R</i>	<i>B</i>

Example: a 3-Coloring of $G(10,10)$

EXAMPLE: A 3-Coloring of $G_{10,10}$

R	R	R	R	B	B	G	G	B	G
R	B	B	G	R	R	R	G	G	B
G	R	B	G	R	B	B	R	R	G
G	B	R	B	B	R	G	R	G	R
R	B	G	G	G	B	G	B	R	R
G	R	B	B	G	G	R	B	B	R
B	G	R	B	G	B	R	G	R	B
B	B	G	R	R	G	B	G	B	R
G	G	G	R	B	R	B	B	R	B
B	G	B	R	B	G	R	R	G	G

It is known that CANNOT 2-color $G_{10,10}$. Hence $\chi(G_{10,10}) = 3$.

Obstruction Sets

Fenner-Gasarch-Glover-Purewall [FGGP] showed:

1. For all c there exists OBS_c , a finite set of grids, such that

$G_{n,m}$ is c -colorable iff no element of OBS_c is inside $G_{m,n}$.

2. FGGP have a proof which shows $|OBS_c| \leq 2c^2$.
3. If OBS_c is known then the set of c -colorable grids is completely characterized.

OBS-2 and OBS-3 Known

FGGP showed

$$OBS_2 = \{G_{3,7}, G_{5,5}, G_{7,3}\}$$

$$OBS_3 = \{G_{4,19}, G_{5,16}, G_{7,13}, G_{10,11}, G_{11,10}, G_{13,7}, G_{16,5}, G_{19,4}\}$$

2-colorability table. *C* for Colorable, *U* for Uncolorability.

	2	3	4	5	6	7
2	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>
3	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>U</i>
4	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>U</i>
5	<i>C</i>	<i>C</i>	<i>C</i>	<i>U</i>	<i>U</i>	<i>U</i>
6	<i>C</i>	<i>C</i>	<i>C</i>	<i>U</i>	<i>U</i>	<i>U</i>
7	<i>C</i>	<i>U</i>	<i>U</i>	<i>U</i>	<i>U</i>	<i>U</i>

4-Colorability

1. FGGP did not (as of 2009) determine OBS_4 .
2. FGGP had reasons to think $G_{17,17}$ is 4-colorable but they did not have a 4-coloring.
3. In 2009 Gasarch offered a prize of \$289.00 for the first person to email him a 4-coloring of $G_{17,17}$.
4. Brian Hayes, Scientific American Math Editor, popularized the challenge.

Challenge Was Hard

1. Lots of people worked on it.
2. No progress.
3. Finally solved in 2012 by Bernd Steinbach and Christian Posthoff [SP]. Clever, and SAT-solver, but did not generalize.
4. They and others also found colorings that lead to $\text{OBS}_4 = \{$

$G_{5,41}, G_{6,31}, G_{7,29}, G_{9,25}, G_{18,23}, G_{11,22}, G_{13,21}, G_{17,19},$
 $G_{41,5}, G_{31,6}, G_{29,7}, G_{25,9}, G_{23,18}, G_{22,11}, G_{21,13}, G_{19,17}$

$\}$

Is Grid Coloring Hard?

We view this two ways:

1. Is there an NP-complete problem lurking here somewhere?
YES!
2. Is there a Prop Statement about Grid Coloring whose resolution proof requires exp size? YES!

THERE IS AN NP-COMPLETE PROBLEM
LURKING!

Grid Coloring Hard!-NP stuff

1. Let $c, N, M \in \mathbb{N}$. A partial mapping χ of $N \times M$ to $\{1, \dots, c\}$ is *extendable to a c -coloring* if there is an extension of χ to a total mapping which is a c -coloring of $N \times M$.
- 2.

$$GCE = \{(N, M, c, \chi) \mid \chi \text{ is extendable}\}.$$

GCE is NP-complete!

GCE is NP-complete

$\phi(x_1, \dots, x_n) = C_1 \wedge \dots \wedge C_m$ is a 3-CNF formula. We determine N, M, c and a partial c -coloring χ of $N \times M$ such that

$$\phi \in 3\text{-SAT} \text{ iff } (N, M, c, \chi) \in GCE$$

Forcing a Color to Only Appear Once in Main Grid

<i>G</i>								<i>G</i>	
<i>G</i>									
<i>R</i>		<i>G</i>							
<i>G</i>									
<i>G</i>									
<i>G</i>									
<i>G</i>	<i>G</i>	<i>G</i>	<i>G</i>	<i>G</i>	<i>G</i>	<i>G</i>	<i>G</i>	<i>G</i>	<i>G</i>

G can only appear once in the main grid, but what about *R*? (The double lines are not part of the construction. They are there to separate the main grid from the rest.)

Forcing a Color to Only Appear Once in Main Grid

<i>R</i>	<i>G</i>								
<i>R</i>	<i>G</i>								
<i>R</i>	<i>R</i>		<i>G</i>						
<i>R</i>	<i>G</i>								
<i>R</i>	<i>G</i>								
<i>R</i>	<i>G</i>								
<i>R</i>	<i>G</i>	<i>G</i>	<i>G</i>	<i>G</i>	<i>G</i>	<i>G</i>	<i>G</i>	<i>G</i>	<i>G</i>
<i>R</i>	<i>G</i>	<i>R</i>	<i>R</i>	<i>R</i>	<i>R</i>	<i>R</i>	<i>R</i>	<i>R</i>	<i>R</i>

Neither *G* nor *R* can be in the main grid.

Using Variables

D means that the color is some *distinct*, unique color.

	D	D	D	D	D	D	D	D	D	D	D
\bar{x}_1		D	D	D	D	D	D	D	D	T	F
x_1		D	D	D	D	D	D	T	F	T	F
\bar{x}_1		D	D	D	D	T	F	T	F	D	D
x_1		D	D	T	F	T	F	D	D	D	D
\bar{x}_1		T	F	T	F	D	D	D	D	D	D
x_1		T	F	D	D	D	D	D	D	D	D

The labeled x_1, \bar{x}_1 are not part of the grid. They are visual aids.

Coding a Clause

$C_1 = L_1 \vee L_2 \vee L_3$. Where L_1, L_2, L_3 are literals (vars or their negations).

	...		D	T	T
	\vdots		\vdots	\vdots	\vdots
L_1		...		D	F
	\vdots		\vdots	\vdots	\vdots
L_2		...			
	\vdots		\vdots	\vdots	\vdots
L_3		...		F	D
	\vdots		\vdots	\vdots	\vdots

The L_1, L_2, L_3 are not part of the grid. They are visual aids.

Coding a Clause—More Readable

$$C_1 = L_1 \vee L_2 \vee L_3.$$

		<i>D</i>	<i>T</i>	<i>T</i>
<i>L</i> ₁			<i>D</i>	<i>F</i>
<i>L</i> ₂				
<i>L</i> ₃			<i>F</i>	<i>D</i>

One can show that

- ▶ If put any of TTT, TTF, TFT, FTT, FFT, FTF, TFF in first column then can extend to full coloring.
- ▶ If put FFF in first column then cannot extend to a full coloring.

Example: (F,F,T)

$$C_1 = L_1 \vee L_2 \vee L_3.$$

		<i>D</i>	<i>T</i>	<i>T</i>
<i>L</i> ₁	<i>F</i>		<i>D</i>	<i>F</i>
<i>L</i> ₂	<i>F</i>			*
<i>L</i> ₃	<i>T</i>		<i>F</i>	<i>D</i>

The * is forced to be *T*.

Example: (F,F,T)

$$C_1 = L_1 \vee L_2 \vee L_3.$$

		<i>D</i>	<i>T</i>	<i>T</i>
<i>L</i> ₁	<i>F</i>		<i>D</i>	<i>F</i>
<i>L</i> ₂	<i>F</i>		*	<i>T</i>
<i>L</i> ₃	<i>T</i>		<i>F</i>	<i>D</i>

The * is forced to be *F*.

Example: (F,F,T)

$$C_1 = L_1 \vee L_2 \vee L_3.$$

		<i>D</i>	<i>T</i>	<i>T</i>
<i>L</i> ₁	<i>F</i>		<i>D</i>	<i>F</i>
<i>L</i> ₂	<i>F</i>		<i>F</i>	<i>T</i>
<i>L</i> ₃	<i>T</i>		<i>F</i>	<i>D</i>

Can color all blanks with *T*.

Example: (F,F,T)

$$C_1 = L_1 \vee L_2 \vee L_3.$$

		<i>D</i>	<i>T</i>	<i>T</i>
<i>L</i> ₁	<i>F</i>	<i>T</i>	<i>D</i>	<i>F</i>
<i>L</i> ₂	<i>F</i>	<i>T</i>	<i>F</i>	<i>T</i>
<i>L</i> ₃	<i>T</i>	<i>T</i>	<i>F</i>	<i>D</i>

Can color all blanks with *T*. This is a coloring.

Other Assignments

1. We did (T, F, F) .
2. (F, T, F) , (F, F, T) are similar.
3. (F, T, T) , (T, F, T) , (T, T, F) , (T, T, T) are easier.

Cannot Use (F,F,F)

$C_1 = L_1 \vee L_2 \vee L_3$. Want that (F, F, F) CANNOT be extended to a coloring.

		D	T	T
L_1	F		D	F
L_2	F		*	*
L_3	F		F	D

The *'s are forced to be T .

Cannot Use (F,F,F)

		<i>D</i>	<i>T</i>	<i>T</i>
<i>L</i> ₁	<i>F</i>		<i>D</i>	<i>F</i>
<i>L</i> ₂	<i>F</i>		<i>T</i>	<i>T</i>
<i>L</i> ₃	<i>F</i>		<i>F</i>	<i>D</i>

There is a mono rectangle of *T*'s. NOT a valid coloring!

Put it all Together

Do the above for all variables and all clauses to obtain the result that GRID EXT is NP-complete!

Big Example

$$(x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_2 \vee x_3 \vee x_4) \wedge (\bar{x}_1 \vee \bar{x}_3 \vee \bar{x}_4)$$

												C_1	C_1	C_2	C_2	C_3	C_3
	D	D	D	D	D	D	D	D	D	D	D	T	T	T	T	T	T
\bar{x}_4		D	D	D	D	D	D	D	D	T	F	D	D	D	D	D	F
x_4		D	D	D	D	D	D	D	D	T	F	D	D	D	F	D	D
\bar{x}_3		D	D	D	D	D	D	T	F	D	D	D	D	D	D	D	D
x_3		D	D	D	D	T	F	T	F	D	D	D	D			D	D
\bar{x}_3		D	D	D	D	T	F	D	D	D	D	D	F	D	D		
\bar{x}_2		D	D	T	F	D	D	D	D	D	D	D	D	F	D	D	D
x_2		D	D	T	F	D	D	D	D	D	D			D	D	D	D
\bar{x}_1		T	F	D	D	D	D	D	D	D	D	D	D	D	D	F	D
x_1		T	F	D	D	D	D	D	D	D	D	F	D	D	D	D	D

Does this Explain why the Challenge was Hard?

1. **MAYBE NOT:** GCE is Fixed Parameter Tractable: For fixed c GCE_c is in time $O(N^2M^2 + 2^{O(c^4)})$. But for $c = 4$ this is huge!
2. **MAYBE NOT:** Our result says nothing about the case where the grid is originally all blank.

Open Questions

1. Improve Fixed Parameter Tractable algorithm.
2. NPC results for mono squares? Other shapes?
3. Show that

$$\{(n, m, c) : G_{n,m} \text{ is } c\text{-colorable}\}$$

is hard.

- ▶ If n, m in unary then sparse set, not NPC—New framework for hardness needed.
- ▶ If n, m binary then not in NP.

YOU SAY YOU WANT A RESOLUTION!

Definition

Let $\varphi = C_1 \wedge \cdots \wedge C_L$ be a CNF formula. A **Resolution Proof** that $\varphi \notin SAT$ is a sequence of clauses such that on each line you have either

1. One of the C 's in φ (called an AXIOM).
2. $A \vee B$ if $A \vee x$ and $B \vee \neg x$ were on prior lines. Variable that is **resolved on** is x .
3. The last line has the empty clause.

Example

$$\varphi = x_1 \wedge x_2 \wedge (\neg x_1 \vee \neg x_2)$$

1. x_1 (AXIOM)
2. $\neg x_1 \vee \neg x_2$ (AXIOM)
3. $\neg x_2$ (From lines 1,2, resolve on x_1 .)
4. x_2 (AXIOM)
5. \emptyset (From lines 3,4, resolve on x_2 .)

Resolution is Complete

Definition

Let $\varphi = C_1 \wedge \cdots \wedge C_L$ be a CNF formula on n variables.

1. If there is a Res Proof that $\varphi \notin SAT$ then $\varphi \notin SAT$.

Proof: Any assignment that satisfies φ satisfies any node of the Res Proof including the last node \emptyset .

2. If $\varphi \notin SAT$ then there is a Res Proof that $\varphi \notin SAT$ of size $2^{O(n)}$.

Proof: Form a Decision Tree that has at every node on level i the variable x_i . Right= T and Left= F . A leaf is the first clause that is false with that assignment. **Turn Decision Tree upside down! View nodes as which var to resolve on! This will be Res Proof!** (It will even be Tree Res Proof.)

Another Example

The AND of the following:

1. For $i, j \in \{1, \dots, 5\}$

$$x_{ij1} \vee x_{ij2}.$$

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Interpretation: (i, j) is colored either 1 or 2.

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2. For $i, j, i', j' \in \{1, \dots, 5\}$

$$\neg x_{ij1} \vee \neg x_{i'j'1} \vee \neg x_{ij'1} \vee \neg x_{i'j1}$$

Another Example

The AND of the following:

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Interpretation: (i, j) is colored either 1 or 2.

2. For $i, j, i', j' \in \{1, \dots, 5\}$

$$\neg x_{ij1} \vee \neg x_{i'j'1} \vee \neg x_{ij2} \vee \neg x_{i'j'2}.$$

Interpretation: There is no mono 1-rectangle.

Another Example

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1. For $i, j \in \{1, \dots, 5\}$

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Interpretation: (i, j) is colored either 1 or 2.

2. For $i, j, i', j' \in \{1, \dots, 5\}$

$$\neg x_{ij1} \vee \neg x_{i'j1} \vee \neg x_{ij'1} \vee \neg x_{i'j'1}$$

Interpretation: There is no mono 1-rectangle.

3. For $i, j, i', j' \in \{1, \dots, 5\}$

$$\neg x_{ij2} \vee \neg x_{i'j2} \vee \neg x_{ij'2} \vee \neg x_{i'j'2}$$

Another Example

The AND of the following:

1. For $i, j \in \{1, \dots, 5\}$

$$x_{ij1} \vee x_{ij2}.$$

Interpretation: (i, j) is colored either 1 or 2.

2. For $i, j, i', j' \in \{1, \dots, 5\}$

$$\neg x_{ij1} \vee \neg x_{i'j'1} \vee \neg x_{ij'1} \vee \neg x_{i'j1}$$

Interpretation: There is no mono 1-rectangle.

3. For $i, j, i', j' \in \{1, \dots, 5\}$

$$\neg x_{ij2} \vee \neg x_{i'j'2} \vee \neg x_{ij'2} \vee \neg x_{i'j2}$$

Interpretation: There is no mono 2-rectangle.

Another Example

The AND of the following:

1. For $i, j \in \{1, \dots, 5\}$

$$x_{ij1} \vee x_{ij2}.$$

Interpretation: (i, j) is colored either 1 or 2.

2. For $i, j, i', j' \in \{1, \dots, 5\}$

$$\neg x_{ij1} \vee \neg x_{i'j1} \vee \neg x_{ij'1} \vee \neg x_{i'j'1}$$

Interpretation: There is no mono 1-rectangle.

3. For $i, j, i', j' \in \{1, \dots, 5\}$

$$\neg x_{ij2} \vee \neg x_{i'j2} \vee \neg x_{ij'2} \vee \neg x_{i'j'2}$$

Interpretation: There is no mono 2-rectangle.

We interpret this statement as saying

There is a 2-coloring of $G_{5,5}$.

This statement is known to be false.

GRID(n, m, c)

Definition

Let $n, m, c \in \mathbb{N}$. $GRID(n, m, c)$ is the AND of the following:

1. For $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$,

$$x_{ij1} \vee x_{ij2} \vee \dots \vee x_{ijc}$$

Interpretation: (i, j) is colored either 1 or \dots or c .

2. For $i, i' \in \{1, \dots, n\}$, $j, j' \in \{1, \dots, m\}$, $k \in \{1, \dots, c\}$,

$$\neg x_{ijk} \vee \neg x_{i'jk} \vee \neg x_{ij'k} \vee \neg x_{i'j'k}$$

Interpretation: There is no mono rectangle.

We interpret this statement as saying

There is a c -coloring of $G_{n,m}$.

NOTE: $GRID(n, m, c)$ has nmc VARS and $O(cn^2m^2)$ CLAUSES.

GRID(n,m,c)—How to View Assignments

Given an assignment:

1. For all $i \in [n]$ and $j \in [m]$ let k be the LEAST number such that $x_{ijk} = T$. View this as saying that (i, j) is colored k .
2. If there is NO such number then (i, j) is not colored and this assignment makes $GRID(n, m, c)$ FALSE.

Hence we view assignments as attempted colorings of the grid where some points are not colored.

Two Ways to Invalidate GRID(n,m,c)

1. There is a mono rectangle.
2. There is some point that is not colored: there is some i,j such that all x_{ijk} are FALSE.

Tree Resolution Proofs

Definition

A **Tree Res Proof** is a Res Proof where the underlying graph is a tree. Note that if you remove the bottom node that is labeled \emptyset then the Tree Res Proof is cut into two **disjoint** parts.

Known: If $\phi \notin SAT$ and ϕ has v variables then there is a Tree Res Proof of ϕ of size $2^{O(v)}$.

Our Goal

Assume that there is no c -coloring of $G_{n,m}$.

1. $GRID(n, m, c)$ has a size $2^{O(cnm)}$ Tree Res Proof.
2. We show $2^{\Omega(c)}$ size is **REQUIRED**. THIS IS OUR POINT!
3. The lower bound is IND of n, m .
4. Example: To prove $GRID(c^3, c^3, c) \notin SAT$ requires Tree Res Proof of size $2^{\Omega(c)}$.

The Prover-Delayer Game

(Due to Beyersdorr, Galesi, Lauria [BGL].) Parameters of the game: $a, b \in (1, \infty)$, with $\frac{1}{a} + \frac{1}{b} = 1$, $p \in \mathbb{R}^+$,

$$\varphi = C_1 \wedge \cdots \wedge C_L \notin \text{SAT}.$$

Do the following until a clause is proven false:

1. **PROVER** picks a variable x that was not already picked.
2. **DEL** either
 - 2.1 Sets x to F or T , OR
 - 2.2 Defers to **PROVER**.
 - 2.2.1 If **PROVER** sets $x = F$ then **DEL** gets $\lg a$ pts.
 - 2.2.2 If **PROVER** sets $x = T$ then **DEL** gets $\lg b$ pts.

At end if **DEL** has $\geq p$ pts then he **WINS**; else **PROVER WINS**.

Convention

We assume that **PROVER** and **DEL** play perfectly.

1. **PROVER** wins means *PROVER has a winning strategy.*
2. **DEL** wins means *DEL has a winning strategy.*

Prover-Del Game and Tree Res Proofs

Lemma

Let $a, b \in (1, \infty)$, such that $\frac{1}{a} + \frac{1}{b} = 1$, $p \in \mathbb{R}^+$, $\varphi \notin \text{SAT}$. If φ has a Tree Res Proof of size $< 2^p$ then **PROVER** wins.

Prover-Del Game and Tree Res Proofs

Lemma

Let $a, b \in (1, \infty)$, such that $\frac{1}{a} + \frac{1}{b} = 1$, $p \in \mathbb{R}^+$, $\varphi \notin \text{SAT}$. If φ has a Tree Res Proof of size $< 2^p$ then **PROVER** wins.

Proof.

We do case $a = b = 2$. **PROVER** Strategy:

1. Initially T is res tree of size $< 2^p$ and **DEL** has 0 pts.
2. **PROVER** picks x , the LAST var **resolved on**.
3. If **DEL** sets x then **DEL** gets no pts.
4. If **DEL** defers then **PROVER** sets T or F —**whichever yields a smaller tree**. **NOTE**: One of the trees will be of size $< 2^{p-1}$. **DEL** gets 1 point.
5. Repeat: after i th stage will always have T of size $< 2^{p-i}$, and **DEL** has $\leq i$ pts. □

Contrapositive is Awesome!

Recall:

Lemma

Let $a, b \in (1, \infty)$, with $\frac{1}{a} + \frac{1}{b} = 1$, $p \in \mathbb{R}^+$, $\varphi \notin \text{SAT}$. If φ has a Tree Res Proof of size $< 2^p$ then **PROVER** wins.

Contrapositive is Awesome!

Recall:

Lemma

Let $a, b \in (1, \infty)$, with $\frac{1}{a} + \frac{1}{b} = 1$, $p \in \mathbb{R}^+$, $\varphi \notin \text{SAT}$. If φ has a Tree Res Proof of size $< 2^p$ then **PROVER** wins.

Contrapositive:

Lemma

Let $a, b \in (1, \infty)$, $\frac{1}{a} + \frac{1}{b} = 1$, $p \in \mathbb{R}^+$, $\varphi \notin \text{SAT}$. If **DEL** wins then **EVERY** Tree Res Proof for φ has size $\geq 2^p$.

PLAN: Get AWESOME strategy for **DEL** when $\varphi = \text{GRID}(n, m, c)$.

GRID(n,m,c) Requires Exp Tree Res Proofs

Theorem

Let n, m, c be such that $G_{n,m}$ is not c -colorable. Let $c \geq 2$. Any tree resolution proof of $GRID(n, m, c) \notin SAT$ requires size $2^{0.5c}$.

PROOF: Parameters: $a, b \in (1, \infty)$ with $\frac{1}{a} + \frac{1}{b} = 1$. We will take $a = b = 2$.

Del Strategy

Assume x_{ijk} was chosen by the **PROVER**.

1. If setting $x_{ijk} = T$ creates a mono rect (of color k) then **DEL** DOES NOT let this happen—he sets x_{ijk} to F .
2. If none of the x_{ij*} are T and $\geq \frac{\epsilon}{2}$ of the x_{ij*} are F via **PROVER** then **DEL** sets x_{ijk} to T .
3. In all other cases the **DEL** defers to the **PROVER**.

Case 1: Prover Set $c/2$ Vars to F

Game ends when there is some i, j such that

$$x_{ij1} = x_{ij2} = \cdots = x_{ijc} = F.$$

Who set those variables to F ?

Case 1: At least $\frac{c}{2}$ set F by **PROVER**. Then **DEL** gets at least

$$0.5c \lg a = 0.5c \text{ pts.}$$

Case 2: Del Set $c/2$ Vars to F

$x_{ij1} = x_{ij2} = \dots = x_{ijc} = F$. Who set those vars to F ?

Case 2: At least $\frac{c}{2}$ set F by **DEL**. Say x_{ijR} set to F by **DEL**. Why?

	j'		j
i'	R		R
i	R		

So $x_{i'jR} = x_{ij'R} = x_{i'j'R} = T$.

For every k such that x_{ijk} is set to F by **DEL** there exists **THREE** vars of form x_{**k} set to T .

KEY: All these 3-sets are **DISJOINT**, so at least $3c/2$ vars set T (by who?).

Case 2a: Prover Set $3c/2$ Vars to T

KEY: At least $3c/2$ vars set T (by who?).

Case 2a: **PROVER** set $\geq \frac{3c}{4}$ to T . **DEL** gets at least

$$0.75c \lg b = 0.75c \text{ pts.}$$

Case 2b: Del Set $3c/2$ Vars To T

Case 2b: DEL set $\geq \frac{3c}{4}$ to T .

DEL set x_{ijk} to T :

- ▶ At time there are $c/2$ k' such that PROVER set $x_{ijk'}$ to F .
- ▶ DEL will NEVER set an x_{ij*} to T again! NEVER!!

Every x_{ijk} set T by DEL implies that $c/2$ vars set F by PROVER, and these sets of $c/2$ vars are disjoint.

UPSHOT: PROVER had set $\frac{3c}{4} \times \frac{c}{2}$ to F . DEL gets at least

$$0.375c^2 \lg a = 0.375c^2 \text{ pts.}$$

Final Analysis

- ▶ **Case 1:** DEL gets at least $0.5c$ pts.
- ▶ **Case 2a:** DEL gets at least $0.75c$ pts.
- ▶ **Case 2b:** DEL gets at least $0.375c^2$ pts.

UPSHOT: For $c \geq 2$ DEL gets at least $2^{0.5c}$ pts.

PUNCHLINE: By **Lemma** any Tree Res Proof has size $\geq 2^{0.5c}$.

Optimize

We can optimize by parametrizing construction and analysis.

1. In construction use cutoff of $c/2$ for when DEL sets x_{ijk} to T . Choose fraction CAREFULLY!
2. In analysis we twice do a half-half cutoff. Choose fractions CAREFULLY!
3. Choose a, b CAREFULLY!

Theorem

Let n, m, c be such that $G_{n,m}$ is not c -colorable. Let $c \geq 9288$. Any tree resolution proof of $GRID(n, m, c) \notin SAT$ requires size $2^{0.836c}$.

Other Shapes

What is special about rectangles? NOTHING!

Definition

(Informally) Let S be a set of at least 2 grid points. Let $GRID(n, m, c; S)$ be the prop statement that there is a c -coloring of $G_{n,m}$ with no mono configuration that is “like S ”.

Theorem

(Informally) Let S be a set of at least 2 grid points. Let n, m, c be such that $GRID(n, m, c; S) \notin SAT$. Any tree resolution proof of $GRID(n, m, c; S) \notin SAT$ requires size $2^{\Omega(c)}$.

Open Questions

1. Want matching upper bounds for Tree Res Proofs that $GRID(n, m, c) \notin SAT$.
2. Want lower bounds on Gen Res Proofs that $GRID(n, m, c) \notin SAT$.
3. Want lower bounds on in other proof systems $GRID(n, m, c) \notin SAT$.
4. Upper and lower bounds for $GRID(n, m, c; S)$ for various S in various proof systems.

- BGL** O. Beyersdorr, N. Galesi, and M. Lauria. A lower bound for the pigeonhole principle in the tree-like resolution asymmetric prover-delayer games. *Information Processing Letters*, 110, 2010. The paper and a talk on it are here:
<http://www.cs.umd.edu/~gasarch/resolution.html>.
- FGGP** S. Fenner, W. Gasarch, C. Glover, and S. Purewal. Rectangle free colorings of grids, 2009.
<http://www.cs.umd.edu/~gasarch/papers/papers.html>.
- SP** B. Steinbach and C. Posthoff. Extremely complex 4-colored rectangle-free grids: Solution of an open multiple-valued problem. In *Proceedings of the Forty-Second IEEE International Symposia on Multiple-Valued Logic*, 2012.
http://www.informatik.tu-freiberg.de/index.php?option=com_content&task=view&id=35&Itemid=63.