On Some Conjectures in Proof Complexity

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Overview

1. Motivation: feasible incompleteness, axiomatic appraoch
2. Syntactically vs. semantically defined classes, deterministic vs. nondeterministic conjectures
3. Finite Σ_1^b reflection principle
4. Herbrand Consistency Search
5. Universal and existential conjectures
6. Problems

Feasible Incompleteness

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1. sentence and propositions speak about finite objects,
2. proofs of them are short and easily constructable.

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"finite version of the 2nd incompleteness theorem"

Gödel, Kreisel, H. Friedman, Cook

basic examples

S, T first order theories, finitely axiomatized, containing a fragment of arithmetic so that they are Σ-complete.

$Con_T(n) :=$ no contradiction of length $\leq n$ in T

Conjecture

If T is sufficiently stronger than S and S is consistent, then S-proofs of $Con_T(n)$ do not have polynomial length (in n).

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Syntactically and semantically defined classes

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2. **NP** $\cap$ **coNP** semantically; probably does not have
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- One theory (e.g. ZFC) does not suffice to prove "$X \in \mathcal{C}$" for all X in $\mathcal{C}$.

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- To show that a language X is in a semantically defined class $\mathcal{C}$, we need a proof.
- One theory (e.g. ZFC) does not suffice to prove "$X \in \mathcal{C}$" for all X in $\mathcal{C}$. ($\Leftrightarrow$ X is not a syntactical class)

Deterministic vs. nondeterministic

Conjecture (deterministic)

If T is sufficiently stronger than S and S is consistent, then S-proofs of $Con_T(n)$ cannot be constructed in polynomial time.

(equivalent to the non-existence of a p-optimal proof system for **TAUT**)

Conjecture (nondeterministic)

If T is sufficiently stronger than S and S is consistent, then S-proofs of $Con_T(n)$ do not have polynomial length.

(equivalent to the non-existence of a length-optimal proof system for **TAUT**)

Conjecture

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Nondeterministic conjectures are stronger.

Reflection principles

Let T be an (arithmetical) theory and ϕ a sentence. Then $Rfn_T(\phi)$ denotes the sentence

$$\forall u(Pr_T(u, \lceil \phi \rceil) \rightarrow \phi),$$

where $Pr_T(u, \lceil \phi \rceil)$ says that u is a proof of ϕ in T and $\lceil \phi \rceil$ is the Gödel number of ϕ.

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Fact $Rfn_T(0 = 1)$ is Con_T.

Let $\sigma(y)$ be a universal Σ_1 formula.

$$\Sigma_1 RFN_T \ := \ \forall u, y(Pr_T(u, \lceil \sigma(\bar{y}) \rceil) \rightarrow \sigma(y))$$

the uniform Σ_1-reflection principle.

The uniform Σ_1^b-reflection principle

$$\Sigma_1^b RFN_T \ := \ \forall u, y(Pr_T(u, \lceil \sigma'(\bar{y}) \rceil) \rightarrow \sigma'(y))$$

where $\sigma'(y)$ be a universal Σ_1^b formula.

Finite Σ_1^b reflection principle

Definition

Let T be a theory, $A(x, y) \in L(PV)$ (i.e., A represents a binary relation decidable in polynomial time), $n \in \mathbb{N}$. Then $\Sigma_1^b Rfn_T^A(n)$ will denote the sentence:

$$\forall u, |u| \leq \bar{n}\, \forall x, |x| \leq \bar{n}\ (Pr_T(u, \lceil \exists y, |y| \leq \bar{n}\ A(\bar{x}, y) \rceil) \rightarrow \exists z, |z| \leq \bar{n}\ A(x, z))$$

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$\Sigma_1^b RFN_T(n)$ will denote $\Sigma_1^b Rfn_T^A(n)$ for a fixed universal polynomial time relation A.

Conjecture

There is no consistent theory S such that for every consistent theory T, proofs of $\Sigma_1^b RFN_T(n)$ in S can be constructed in polynomial time.

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Conjecture (nondeterministic version)

There is no consistent theory S such that for every consistent theory T, there are proofs of $\Sigma_1^b RFN_T(n)$ in S of polynomial length.

Proposition

If **P=NP**, *then there exists a consistent theory* S *such that for every consistent theory* T *and every* $B \in L(PV)$, *proofs of* $\Sigma^b_1 Rfn^B_T(n)$ *in* S *can be constructed in polynomial time.*

Proof.

Note that

- If T is consistent, then the sentences $\Sigma^b_1 Rfn^B_T(n)$ are true.
- If **P=NP**, then we can test in polynomial time whether $\Sigma^b_1 Rfn^B_T(n)$ is true.

So we axiomatize S by the true sentences of this form.

(One can also formalize it by a finite number of axioms.) □

Proof systems for **SAT**

A proof system for **SAT** is an $A(x, y) \in L(PV)$ which is sound and complete when interpreted as "y is a proof of $x \in$ **SAT**".

A polynomial simulation of A by B is an $f \in$ **FP** such that $A(x, y) \rightarrow B(x, f(x, y))$.

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Examples

1. $sat(x, y)$ — the standard proof system for **SAT**, where y is a satisfying assignment for x.
2. $sat(x, y)$, or y is a witness (using [AKS]) that n is not a prime if x is the proposition "n is not a prime".

If factoring is hard, 2. is not reducible to 1.

Proposition

Suppose that S is a consistent theory such that for every consistent theory T and every $B \in L(PV)$, proofs of $\Sigma_1^b Rfn_T^B(n)$ in S can be constructed in polynomial time.
Then there exists a p-optimal proof system for **SAT**.

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Then there exists a p-optimal proof system for **SAT**.

Proof.

Given S, define a proof system for **SAT** by:

- y is a proof of $x \Leftrightarrow y$ is an S proof of $\exists z\ sat(\bar{x}, z)$.

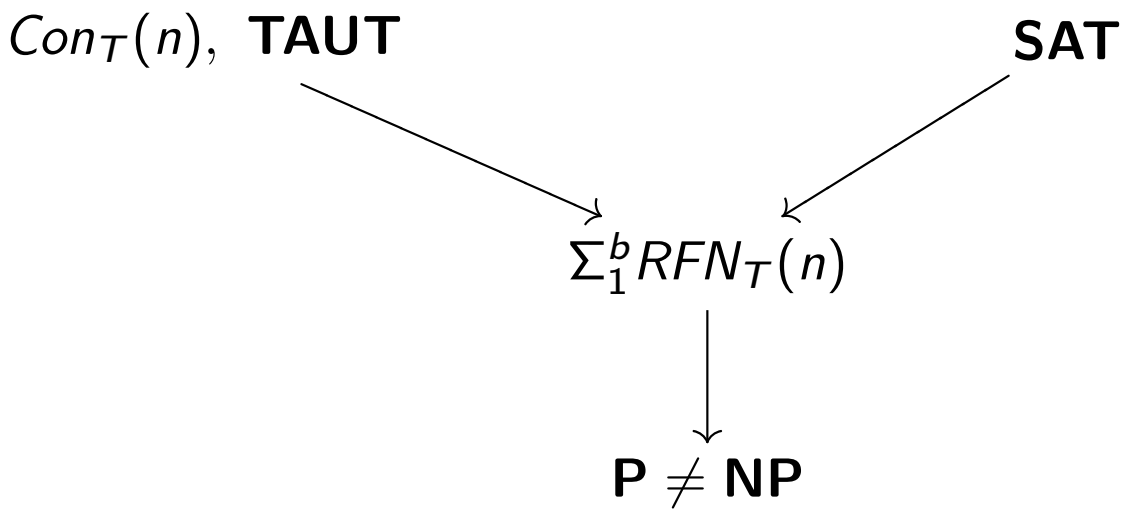
If A is a proof system for **SAT**, then let T be axiomatized by

$$\forall x, y(A(x, y) \rightarrow \exists z\ sat(x, z)).$$

Given an A-proof of ϕ, we can construct a T-proof of

$$\exists z\ sat(\bar{\phi}, z),$$

from which we get an S-proof of the same using $\Sigma^b_1 Rfn^B_T(n)$ with $n = |\phi|$ and $B(x, y) = sat(x, y)$. □



legend:

- $Con_T(n)$ — for no consistent S, we can construct S-proofs of $Con_T(n)$ in poly. time
- $\Sigma_1^b RFN_T(n)$ — for no consistent S, we can construct S-proofs of $\Sigma_1^b RFN_T(n)$ in poly. time
- **TAUT** — no optimal proof system for **TAUT**
- **SAT** — no optimal proof system for **SAT**

Total Polynomial Search

Definition

TPS $:= \{R \in \mathbf{P} \ ; \ \forall x \exists y, |y| \leq |x| \ R(x,y)\}$

Other notation **TFNP**, $\mathbf{NPMV_t}$.

R is reducible to Q if there are $f, g \in \mathbf{FP}$ such that for all x and z,

$$Q(f(x), z) \ \Rightarrow \ R(x, g(x,z)).$$

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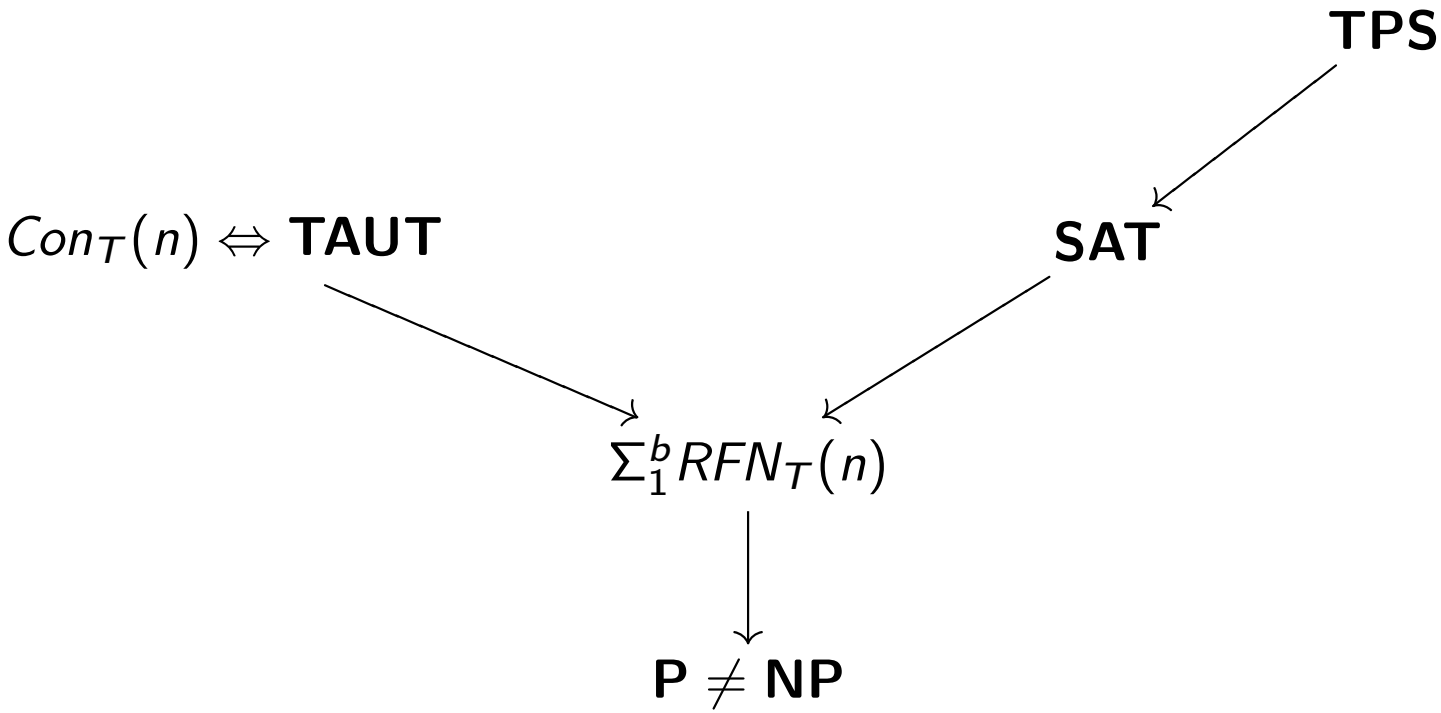
Conjecture

There is no complete **TPS** *problem.*

Essentially, it means that **TPS** is not syntactical.



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- $Con_T(n)$ ($\Sigma_1^b RFN_T(n)$) — for no consistent S we can construct S-proofs of $Con_T(n)$ ($\Sigma_1^b RFN_T(n)$) in poly. time
- **TAUT** (**SAT**) — no optimal proof system for **TAUT** (**SAT**)
- **TPS** — no complete total poly. search problem problem

Herbrand Consistency Search

Definition

Let Φ be $\forall x_1 \ldots \forall x_k \phi(x_1, \ldots, x_k)$ where ϕ is open. Then $HCS(\Phi)$, the Herbrand Consistency Search for Φ, is the following search problem:

- given terms τ_{ij} in the language of ϕ, $i = 1, \ldots, n$, $j = 1, \ldots, k$, find a truth assignment to the atomic subformulas of $\phi(\tau_{i1}, \ldots, \tau_{ik})$, for $i = 1, \ldots, n$, that makes $\bigwedge_{i=1}^{n} \phi(\tau_{i1}, \ldots, \tau_{ik})$ true.[1]

If Φ is consistent, then $HCS(\Phi) \in \mathbf{TPS}$.

[1] which shows that $\bigvee_{i=1}^{n} \neg\phi(\tau_{i1}, \ldots, \tau_{ik})$ is not a Herbrand proof of $\neg\Phi$.

Examples

1. Skolemization of the theory of dense linear orderings:

$$0 < 1$$

$$\neg x < x,\ x < y \vee x = y \vee y < x,$$

$$x < y \wedge y < z \rightarrow x < z,$$

$$x < f(x, y) \wedge f(x, y) < y,$$

plus the identity and equality axioms. To solve the HCS, we need only to find an interpretation of the terms in a finite linear ordering and then to assign the truth values according to this interpretation.

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2. Skolemization of a strong finite fragment of arithmetic. The Skolem functions are not computable, hence interpretation in $\mathbb{N}$ does not help us.

Probably not solvable in polynomial time.

Proposition

If $\Phi \vdash \Psi$, *then* $HCS(\Psi)$ *is reducible to* $HCS(\Phi)$.

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Theorem

For every total polynomial search problem R, *there exist a consistent universal sentence* Φ *such that the problem* R *is reducible to* $HCS(\Phi)$.

Proof-idea.

Given $R \in \mathbf{TPS}$, formalize the sentence $\forall x \exists y, |y| \leq |x|\ R(x, y)$ in a suitable way. □

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The conjecture about the nonexistence of a complete problem in **TPS** is equivalent to:

Conjecture

The complexity of $HCS(\Phi)$ *increases with the increasing strength of* Φ.

Disjoint pairs

Definition

- Disjoint **NP** pairs,
  $Disj\mathbf{NP} := \{(X, Y)\ ;\ X, Y \in \mathbf{NP}\ \wedge\ X \cap Y = \emptyset\}$;
- Disjoint **coNP** pairs,
  $Disj\mathbf{coNP} := \{(X, Y)\ ;\ X, Y \in \mathbf{coNP}\ \wedge\ X \cap Y = \emptyset\}$

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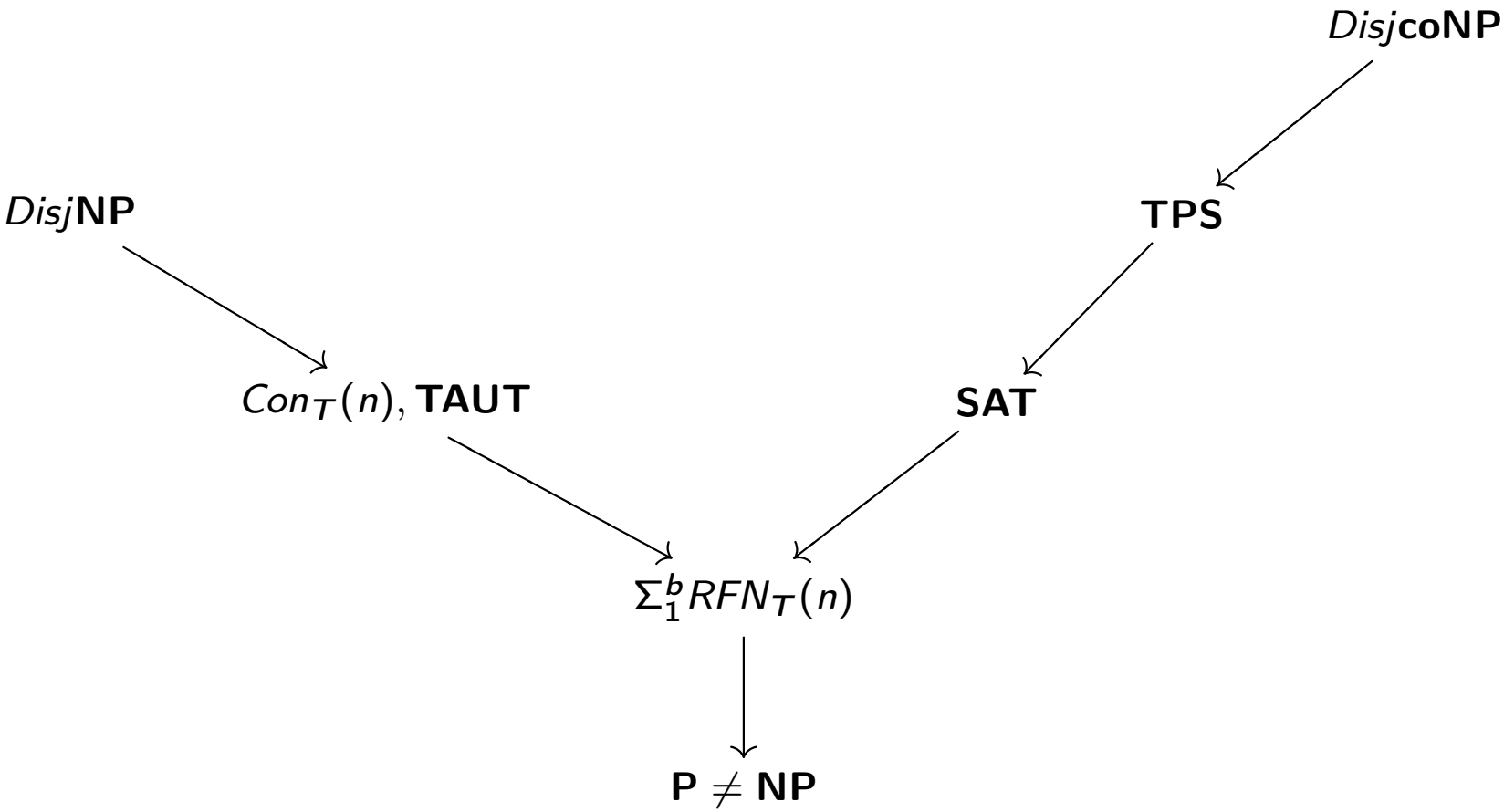
Conjecture

*There is no complete pair in Disj***NP**.

Conjecture

*There is no complete pair in Disj***coNP**.



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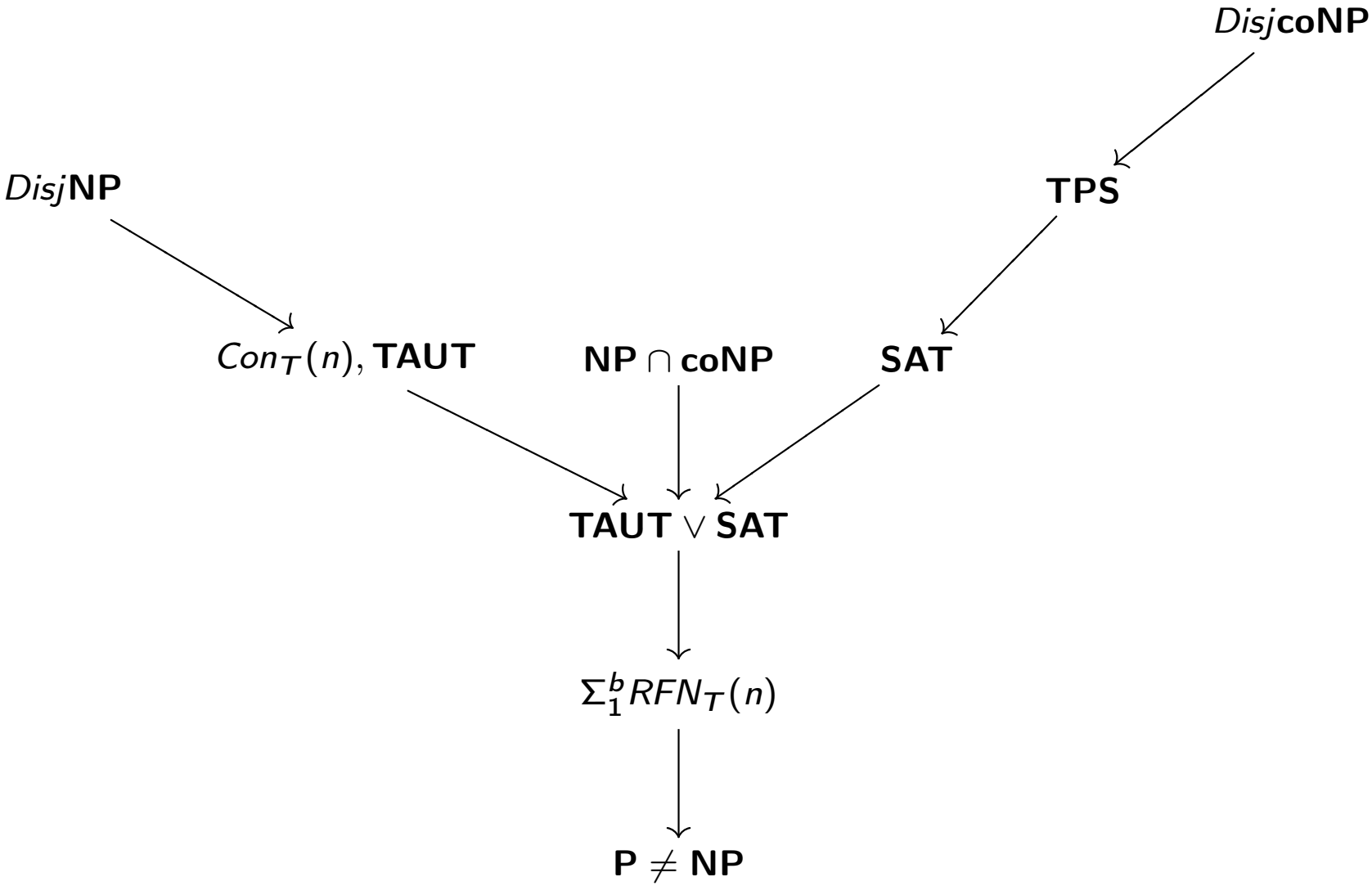
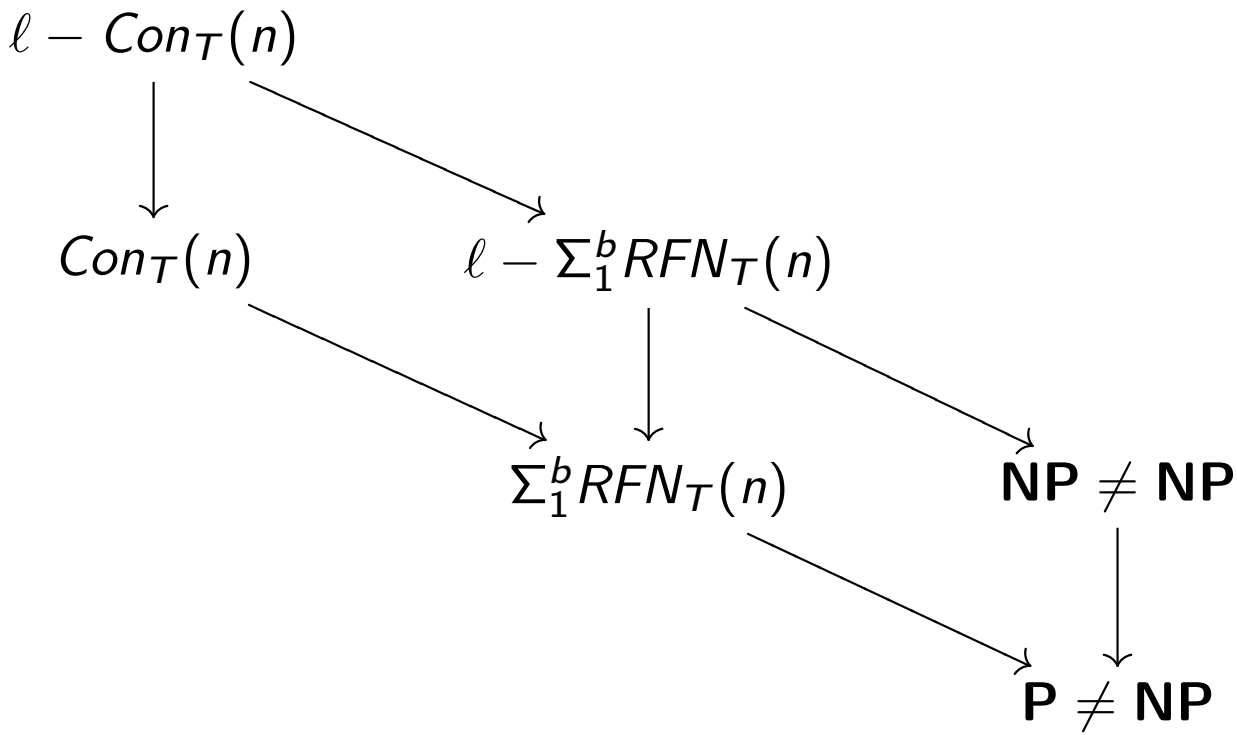
- $Disj$**NP** ($Disj$**coNP**) — no complete $Disj$**NP** ($Disj$**coNP**) pair



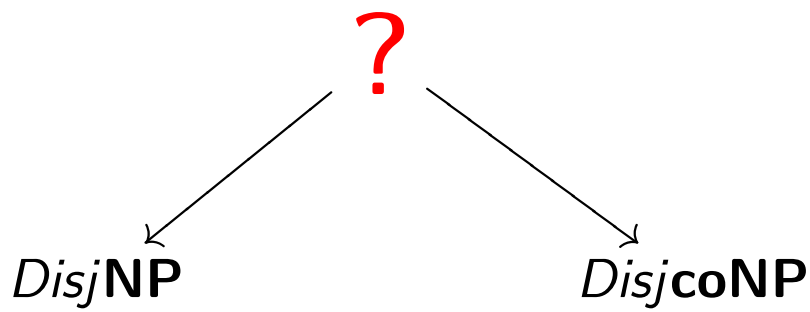
legend:

- **NP** ∩ **coNP** – no complete set in **NP** ∩ **coNP**

Nondeterministic conjectures



Problems

- 

- connections with other open problems/conjectures in computational complexity; in particular noncollaps of **PH**, cryptographic conjectures etc.
- nondeterministic versions
- relativizations

Thank you!